Phase Transitions and Renormalization Group ❷

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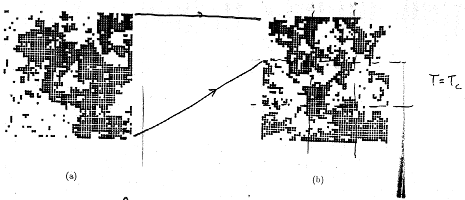
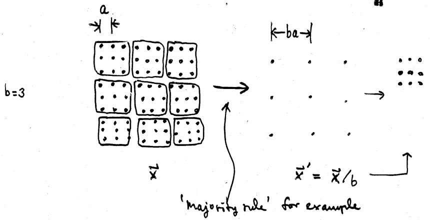
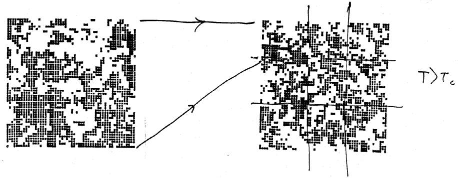
Renormalization Group Transformation

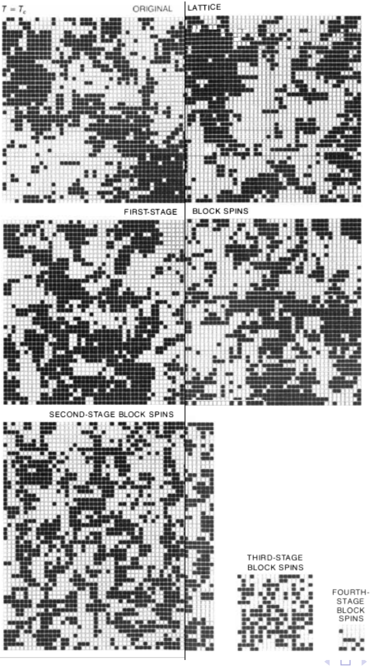
Block Spin transformation

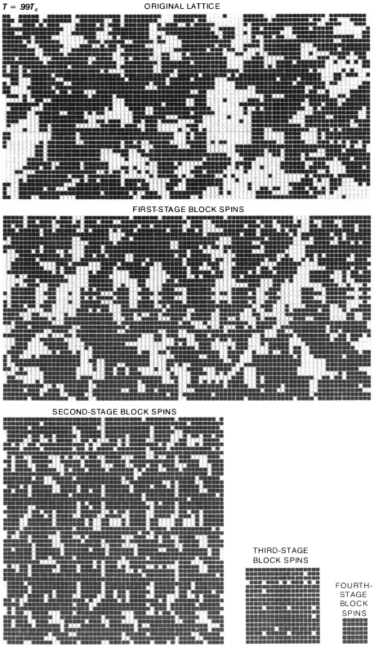







$T = 1.22T_c$
ORIGINAL LATTICE
FIRST-STAGE BLOCK SPINS
SECOND-STAGE BLOCK SPINS
THIRD-STAGE BLOCK SPINS
FOURTH-STAGE BLOCK SPINS


$T = T_c$
ORIGINAL
LATTICE
FIRST-STAGE
BLOCK SPINS
SECOND-STAGE BLOCK SPINS
THIRD-STAGE
BLOCK SPINS
FOURTH-
STAGE
BLOCK
SPINS

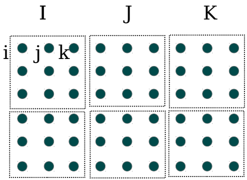

$T = .99T_c$
ORIGINAL LATTICE
FIRST-STAGE BLOCK SPINS
SECOND-STAGE BLOCK SPINS
THIRD-STAGE BLOCK SPINS
FOURTH-STAGE BLOCK SPINS

idea of RG

- Integrate out short distance degrees of freedom:

$$-\mathcal{H} \equiv -\beta H = NK_0 + K_1 \sum_i \sigma_i + K_2 \sum_{i,j} \sigma_i \sigma_j + K_3 \sum_{i,j,k} \sigma_i \sigma_j \sigma_k + \cdots$$

  is described by $\{K\}$
- Block Spins



b=3

$$s_I = \frac{1}{b^d} \sum_{i \in I} \sigma_i, \quad \text{or} \quad s_I = \text{sign}(\sum_{i \in I} \sigma_i)$$

- Projection:

$$P(s_I; \{\sigma_i\}) = \delta(s_I - \text{sign}(\sum_{i \in I} \sigma_i))$$

Note that

$$\sum_{\{s_I\}} \prod_I P(s_I; \{\sigma_i\}) = 1$$

$$Z_N[K] = \sum_{\{\sigma_i\}} e^{-\mathcal{H}[\sigma]} = \sum_{\{\sigma_i\}} \sum_{\{s_I\}} \prod_I P(s_I; \{\sigma_i\}) e^{-\mathcal{H}[\sigma]}$$
$$\equiv \sum_{\{s_I\}} e^{-\mathcal{H}'[s_I]} = Z_{N'}[K'],$$

where $N' = N/b^d$ and

$$\boxed{e^{-\mathcal{H}'[s_I]} \equiv \sum_{\{\sigma_I\}} \prod_I P(s_I; \{\sigma_i\}) e^{-\mathcal{H}[\sigma]}}$$

RG transformation

$$\boxed{\mathcal{R}_b : \ \{K\} \to \{K'\}}$$

In practice, this is difficult to do!

RG / 1d Ising model

Recall that

$$\mathcal{H} = -K \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i,$$

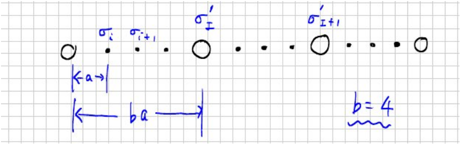
with $K \equiv \beta J$ and $B \equiv \beta h$

$$Z = \sum_{\{\sigma_i = \pm 1\}} T_{\sigma_1 \sigma_2} T_{\sigma_2 \sigma_3} \cdots T_{\sigma_N \sigma_1} = \mathrm{Tr}\, \mathbb{T}^N,$$

➢ Decimation: Trace over $b-1$ spins and leave a spin at every b-th site.

$$Z_N(K,B) = \mathrm{Tr}\, \mathbb{T}^N = \mathrm{Tr}\, (\mathbb{T}^b)^{N'} = Z_{N'}(K', B'),$$

where $N' = N/b$.



❑ $h=0$ Case:
Recall that

$$\begin{aligned} T_{\sigma\sigma'} &= \cosh K(1+\sigma\sigma' \tanh K) \\ \left(T^b\right)_{\sigma\sigma'} &= 2^{b-1}\cosh^b K(1+\sigma\sigma'\tanh^b K) \\ &= \left(\frac{2^{b-1}\cosh^b K}{\cosh K'}\right)\cosh K'(1+\sigma\sigma'\tanh K') \end{aligned}$$

Note that constant term is generated. We may as well start from

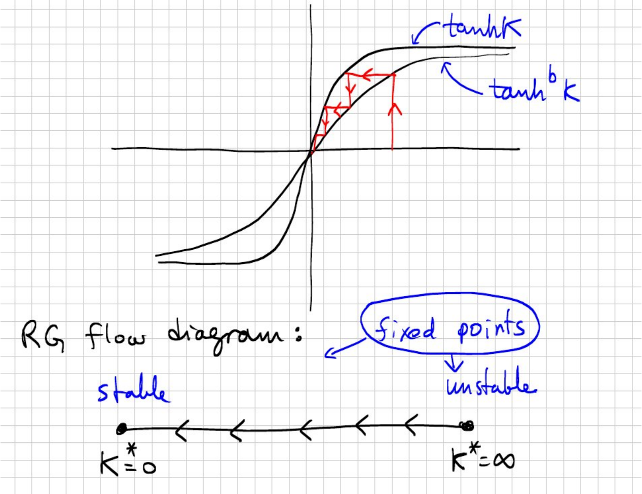
$$-\mathcal{H} = Ng + K\sum_i \sigma_i\sigma_{i+1} \;\to\; -\mathcal{H}' = N'g' + K'\sum_I \sigma_I\sigma_{I+1}$$

$$\begin{aligned} T_{\sigma\sigma'} &= e^g\cosh K(1+\sigma\sigma'\tanh K) \\ \left(T^b\right)_{\sigma\sigma'} &= e^{bg}2^{b-1}\cosh^b K(1+\sigma\sigma'\tanh^b K) \\ &= e^{g'}\cosh K'(1+\sigma\sigma'\tanh K') \end{aligned}$$

We therefore have for $h = 0$ the RG flow equation $(K', g') = \mathcal{R}_b(K, g)$ as

$$g' = bg + \ln\left(\frac{2^{b-1}\cosh^b K}{\cosh K'}\right)$$

$$K' = \tanh^{-1}\left(\tanh^b K\right)$$

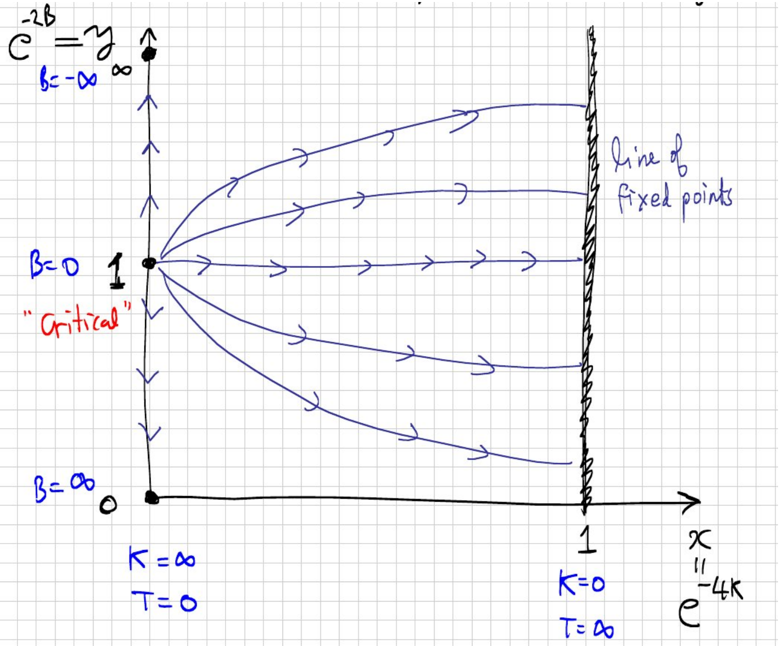


❑ <u>$h \neq 0$ Case</u>: Take $b = 2$ for simplicity

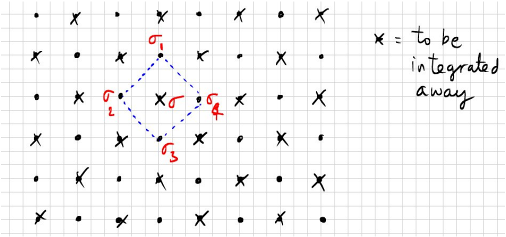
$$\mathbb{T}^2 = e^{2g} \begin{pmatrix} e^{2K+2B} + e^{-2K} & e^{B} + e^{-B} \\ e^{B} + e^{-B} & e^{2K-2B} + e^{-2K} \end{pmatrix}$$
$$\equiv e^{g'} \begin{pmatrix} e^{K'+B'} & e^{-K'} \\ e^{-K'} & e^{K'-B'} \end{pmatrix}$$

$$y' \equiv e^{-2B'} = x \frac{(1+y)^2}{(x+y)(1+xy)}$$
$$x' \equiv e^{-4K'} = y \frac{x+y}{1+xy},$$

where $x \equiv e^{-4K}$ and $y \equiv e^{-2B}$. (Derive these.)


$e^{-2B} = y$
∞
$B=-\infty$
$B=0$
1
"Critical"
$B=\infty$
0
line of
fixed points
1
$K=\infty$
$T=0$
$K=0$
$T=\infty$
$x = e^{-4K}$

decimation in higher dimensions



$$\sum_{\sigma=\pm 1} e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\sigma} = \exp[\ln(2\cosh(K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))]$$

This is not of the form

$$e^{(K'/2)(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)}$$

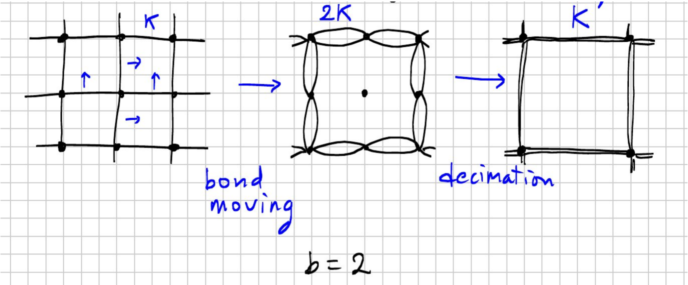
In fact, one can show that this can be written as (show this!)

$$\exp[A' + \frac{K'}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) + L'(\sigma_1\sigma_3 + \sigma_2\sigma_4) + M'\sigma_1\sigma_2\sigma_3\sigma_4]$$

Need an approxmiation scheme.

Migdal-Kadanoff approximation

$\underline{d = 2}$



Let us recall the recursion relation for decimation in 1d ($b = 2$)

$$K' = \tanh^{-1}(\tanh^2 K) = \frac{1}{2} \ln\cosh(2K)$$

From the above figure, we have in this case

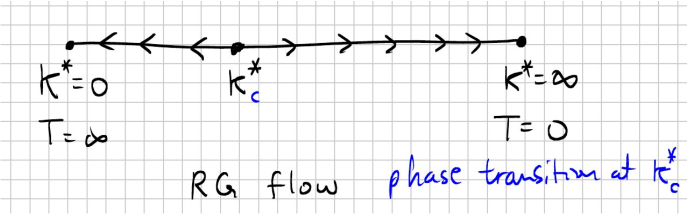
$$\boxed{K' = \frac{1}{2} \ln\cosh(2 \cdot 2K)}$$

Fixed points

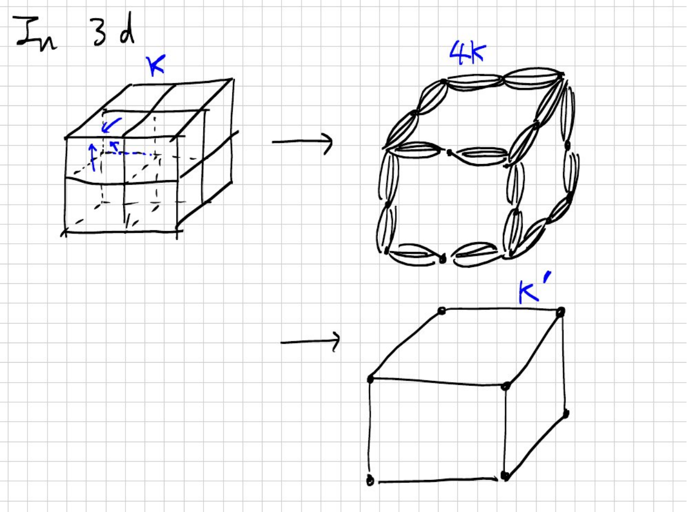
- $K^* = 0$ For $K << 1$, $K' \simeq \frac{1}{2}\ln(1+8K^2) \simeq 4K^2 \rightarrow$ Stable
- $K^* = \infty$ For $K >> 1$, $K' \simeq \frac{1}{2}\ln(e^{4K}/2) \simeq 2K \rightarrow$ Stable
- Critical fixed point

$$e^{2K_c^*} = \frac{1}{2}(e^{4K_c^*} + e^{-4K_c^*})$$

$$K_c^* \simeq 0.305 \qquad (\text{c.f.} \quad K_c^{\text{exact}} \simeq 0.441)$$



$d = 3$



$$K' = \frac{1}{2} \ln \cosh(2 \cdot 4K)$$

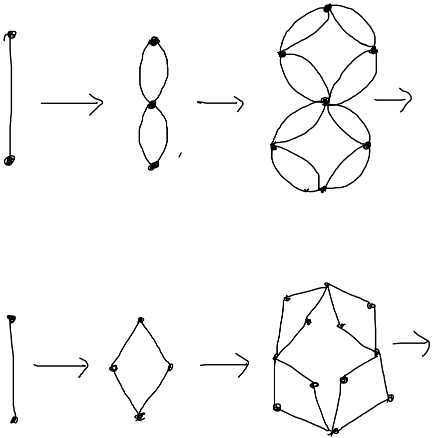
$K_c^* \simeq 0.065 \qquad (\text{c.f.} \quad K_c^{\text{known}} \simeq 0.222)$

In d dimensions,

$$\boxed{K' = \frac{1}{2}\ln\cosh(2\cdot 2^{d-1}K)}$$

MK approximation becomes worse as $d \to \infty$.
It becomes exact on hierarchical lattices

Hierarchical Lattices



For an arbitrary b,

$$\boxed{\tanh K' = \left[\tanh(b^{d-1}K)\right]^{b}}$$

Generalize to continuous $b = 1 + \delta\ell = e^{\delta\ell}$; $K = K(\ell)$

$$\begin{aligned}
\tanh K' &= \left[\tanh((1+\delta\ell)^{d-1}K)\right]^{1+\delta\ell} \\
&\simeq [\tanh((1+(d-1)\delta\ell)K)]^{1+\delta\ell} \\
&\simeq \tanh((1+(d-1)\delta\ell)K)[1+\delta\ell \ln\tanh K] \\
&\simeq \tanh((1+(d-1)\delta\ell)K) + (\delta\ell)\frac{\sinh K \cosh K}{\cosh^2 K}\ln\tanh K \\
&\simeq \tanh\left[K + (d-1)\delta\ell K + (\delta\ell)\frac{1}{2}\sinh(2K)\ln\tanh K\right]
\end{aligned}$$

$$\boxed{\frac{dK}{d\ell} = (d-1)K + \frac{1}{2}\sinh(2K)\ln\tanh K}$$

Ising model in $1 + \varepsilon$ dimensions

Look at large K (or small T) region.

$$\frac{1}{2}\sinh(2K)\ln\tanh K \simeq \frac{1}{2}\frac{e^{2K}}{2}\ln(1-2e^{-2K}) \simeq -\frac{1}{2}$$

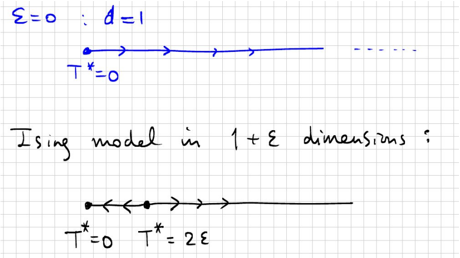
For $d = 1 + \varepsilon$,

$$\frac{dK}{d\ell} \simeq \varepsilon K - \frac{1}{2}$$

or for $T = 1/K$,

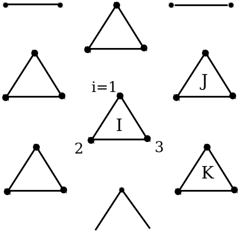
$$\boxed{\frac{dT}{d\ell} \simeq -\varepsilon T + \frac{1}{2}T^2}$$

$d = 1$ is the lower critical dimension of the Ising model.



approximation methods

Niemeijer and van Leeuwen (1974)



- 2-d Ising model on a triangular lattice
- $-\mathcal{H} = \sum_{\langle i,j \rangle} K\sigma_i\sigma_j + h\sum_i \sigma_i$
- Projection (majority rule):

$$P(\mu_I; \sigma_{1I}, \sigma_{2I}, \sigma_{3I}) = P(\mu_I; \{\sigma_I\})$$
$$= \delta(\mu_I, \mathrm{sign}(\sum_i \sigma_{iI}))$$

- Note that

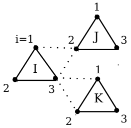
$$\sum_{\mu_I = \pm 1} P(\mu_I; \{\sigma_I\}) = 1$$

$$Z = \sum_{\{\sigma\}} e^{-\mathcal{H}(\{\sigma\},K,h)} = \sum_{\{\sigma\}} \underbrace{\sum_{\{\mu\}} \prod_I P(\mu_I; \{\sigma_I\})}_{=1} \; e^{-\mathcal{H}(\{\sigma\},K,h)}$$

$$= \sum_{\{\mu\}} \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\}) \; e^{-\mathcal{H}(\{\sigma\},K,h)} \equiv \sum_{\{\mu\}} e^{N' K_0'} e^{-\mathcal{H}'(\{\mu\},K',h')}$$

The main difficulty lies in evaluating $\sum_\sigma$ to get $\mathcal{H}'$. Need to use an approximation. Write $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, where

$$-\mathcal{H}_0 = K \sum_I (\sigma_{1I}\sigma_{2I} + \sigma_{2I}\sigma_{3I} + \sigma_{3I}\sigma_{1I})$$
$$+ h \sum_I (\sigma_{1I} + \sigma_{2I} + \sigma_{3I}),$$
$$-\mathcal{V} = K \sum_{\langle I,J \rangle} \sum_{i,j} \sigma_{iI}\sigma_{jJ}, \quad \text{for example,}$$
$$-\mathcal{V}_{IJ} = K(\sigma_{1I}\sigma_{2J} + \sigma_{3I}\sigma_{2J}),$$
$$-\mathcal{V}_{IK} = K(\sigma_{3I}\sigma_{1K} + \sigma_{3I}\sigma_{2K})$$



Let

$$Z_0 = \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\})\ e^{-\mathcal{H}_0(\{\sigma\},K,h)}.$$

Then

$$e^{N'K_0'} e^{-\mathcal{H}'(\{\mu\},K',h')} = Z_0 \langle e^{-\mathcal{V}} \rangle,$$

where

$$\langle A \rangle = \frac{1}{Z_0} \sum_{\{\sigma\}} \prod_I P(\mu_I; \{\sigma_I\})\ A\ e^{-\mathcal{H}_0(\{\sigma\},K,h)}$$

Use a cumulant expansion and take the lowest order as a first approximation.

$$\langle e^{-\mathcal{V}} \rangle = \exp\left[-\langle \mathcal{V} \rangle + \frac{1}{2}(\langle \mathcal{V}^2 \rangle - \langle \mathcal{V} \rangle^2) + \cdots \right] \simeq e^{-\langle \mathcal{V} \rangle}$$

Need to evaluate Z_0 and $\langle \mathcal{V} \rangle$.

- $\mu_I = +1$

σ_{1I}	σ_{2I}	σ_{3I}	$e^{-\mathcal{H}_0}$
$+$	$+$	$+$	e^{3K+3h}
$+$	$+$	$-$	e^{-K+h}
$+$	$-$	$+$	e^{-K+h}
$-$	$+$	$+$	e^{-K+h}

$$\sum_{\sigma_I} e^{-\mathcal{H}_0} = e^{3K+3h} + 3e^{-K+h}$$

- $\mu_I = -1$

σ_{1I}	σ_{2I}	σ_{3I}	$e^{-\mathcal{H}_0}$
$-$	$-$	$-$	e^{3K-3h}
$-$	$-$	$+$	e^{-K-h}
$+$	$+$	$-$	e^{-K-h}
$+$	$-$	$-$	e^{-K-h}

$$\sum_{\sigma_I} e^{-\mathcal{H}_0} = e^{3K-3h} + 3e^{-K-h}$$

The above results can be summarized as $e^{A(K,h)+B(K,h)\mu_I}$ with

$$e^{A+B} = e^{3K+3h} + 3e^{-K+h}, \quad e^{A-B} = e^{3K-3h} + 3e^{-K-h}.$$

For $\langle \mathcal{V} \rangle$, note that in $\mathcal{H}_0$, there is no mixing between σ_{iI} and σ_{jJ} for $I \neq J$. Therefore,

$$-\langle \mathcal{V} \rangle = K \sum_{\langle I,J \rangle} \sum_{i,j} \langle \sigma_{iI} \rangle \langle \sigma_{jJ} \rangle$$

Note that $\langle \sigma_{iI} \rangle$ is indep. of i for given I.

- $\mu_I = +1$

$$\langle \sigma_{iI} \rangle = \frac{e^{3K+3h} + 2e^{-K+h} - e^{-K+h}}{e^{3K+3h} + 3e^{-K+h}} \equiv C(K,h) + D(K,h)$$

- $\mu_I = -1$

$$\langle \sigma_{iI} \rangle = \frac{-e^{3K-3h} - 2e^{-K-h} + e^{-K-h}}{e^{3K-3h} + 3e^{-K-h}} \equiv C(K,h) - D(K,h)$$

This can be summarized as $\langle \sigma_{iI} \rangle = C(K,h) + D(K,h)\mu_I$

We have

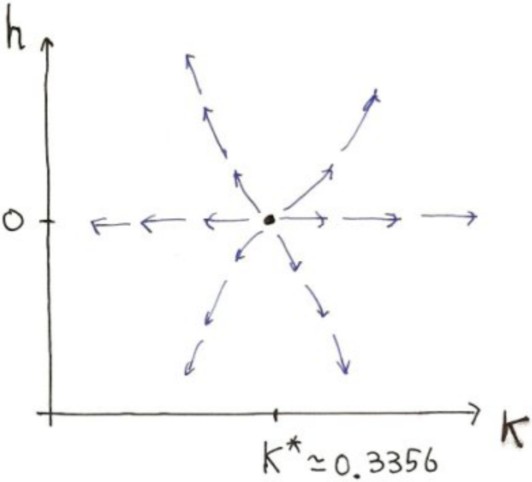
$$Z_0 = \prod_I e^{A+B\mu_I} = e^{N'A+B\sum_I \mu_I}.$$

and

$$\begin{aligned} -\langle \mathcal{V} \rangle &= 2K \sum_{\langle I,J \rangle} (C + D\mu_I)(C + D\mu_J) \\ &= 2K \left[C^2 \frac{N'z}{2} + CDz \sum_I \mu_I + D^2 \sum_{\langle I,J \rangle} \mu_I \mu_J \right], \end{aligned}$$

where $N' = N/3$ and $z = 6$ is the number of nearest neighbors. The factor 2 comes from the two ways of connecting neighboring blocks. We finally have the recursion relations

$$\begin{aligned} K' &= 2KD^2(K,h) \\ h' &= B(K,h) + 12KC(K,h)D(K,h) \\ K_0' &= A(K,h) + 6KC^2(K,h) \end{aligned}$$



$$K_c^{exact} \simeq 0.27465$$

$$K_c^{MF} = \frac{1}{z} = \frac{1}{6}$$

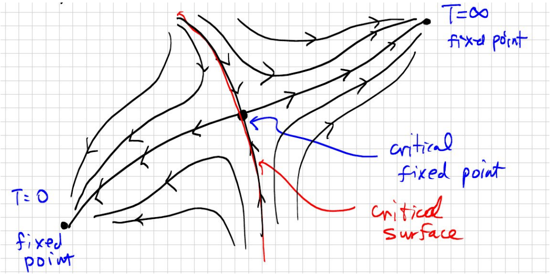
general scaling theory

- RG transform $\mathcal{R}_b : \{K\} \to \{K'\}$ generates a flow in a multi-dimensional parameter space
- Length scale shrinks down at each step

$$\xi(K') = \xi(K)/b$$

- Fixed Points:

$$\mathcal{R}_b\{K^*\} = K^*, \quad \xi(K^*) = \xi(K^*)/b, \quad \xi = 0, \text{ or } \xi = \infty \text{ (critical)}$$



- Linearize $\mathcal{R}_b$ near critical fixed point:

$$K'_a - K^*_a \sim \sum_b T_{ab}(K_b - K^*_b)$$

- Solve eigenvalue equation

$$\sum_a \phi_a^{(i)} T_{ab} = \lambda^{(i)} \phi_b^{(i)}$$

- Multiply the linearized flow eq. by $\phi_a^{(i)}$ and sum over a

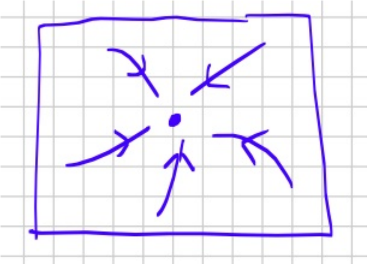
$$\sum_a \phi_a^{(i)}(K'_a - K^*_a) = \lambda^{(i)} \sum_b \phi_b^{(i)}(K_b - K^*_b)$$

or

$$\boxed{u'_i = \lambda^{(i)} u_i},$$

where $u_i \equiv \sum_a \phi_a^{(i)}(K_a - K^*_a)$ is the scaling variable.

- Classification: Let $\lambda^{(i)} \equiv b^{y_i}$ (y_i scaling exponent).
  - Relevant variable: $y_i > 0$ (u_i flows away from the fixed point)
  - Irrelevant variable: $y_i < 0$ ($u_i \to 0$ under RG flow)
  - Marginal variable: $y_i = 0$
- Critical surface: a surface spanned by irrelevant variables



- One has to adjust relevant variables to be on the critical surface.
- Ising universality class: 2 relevant scaling variables
  - 1 thermal, $u_t \sim t \equiv (T - T_c)/T_c$, $y_t > 0$
  - 1 magnetic, $u_h \sim h \equiv \beta h$, $y_h > 0$

scaling theory for Ising universality class

- Free energy per site: $f(\{K\}) = -(1/N)\ln Z(\{K\})$. Invariance of partition function gives

$$Nf(\{K\}) = N'f(\{K'\}) + Ng(\{K\}), \qquad N' = N/b^d$$

where $g(\{K\})$ is a constant term. The singular part behaves as

$$\boxed{f_s(\{K\}) = b^{-d} f_s(\{K'\})}$$

$$f_s(t,h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h) = \cdots = b^{-nd} f_s(b^{n y_t} t, b^{n y_h} h)$$

t and h are growing. Stop at $|b^{n y_t} t| \sim t_0 = O(1)$.

$$\boxed{f_s(t,h) = \left|\frac{t}{t_0}\right|^{d/y_t} \Phi\left(\frac{h}{|t/t_0|^{y_h/y_t}}\right)}$$

- Correlation function: $G(\mathbf{r}_1 - \mathbf{r}_2; t)$

$$\frac{\delta^2 Z}{Z} = \sum_{sites} \delta h(\mathbf{r}_1)\delta h(\mathbf{r}_2) G(\mathbf{r}_1 - \mathbf{r}_2; t) = \sum_{blocks} \delta h'(\mathbf{r}'_1)\delta h'(\mathbf{r}'_2) G(\mathbf{r}'_1 - \mathbf{r}'_2; t')$$

Note that $\delta h'(\mathbf{r}') = b^{y_h}\delta h(\mathbf{r})$, $\mathbf{r}' = \mathbf{r}/b$, and b^d spins in each block.

$$G(\mathbf{r}, t) = b^{2y_h - 2d} G(\mathbf{r}/b, b^{y_t} t) = b^{2n(y_h - d)} G(\mathbf{r}/b^n, b^{ny_t} t)$$

Stop at $|b^{ny_t} t| \sim 1$.

$$\boxed{G(\mathbf{r}, t) = |t|^{\frac{2(d - y_h)}{y_t}} \Psi\left(\frac{\mathbf{r}}{|t|^{-\frac{1}{y_t}}}\right)}$$

- Correlation length $\xi \sim |t|^{-1/y_t} \equiv |t|^{-\nu}$. $\boxed{y_t = 1/\nu}$
- At $T = T_c$ $(t = 0)$, stop at $r/b^n \sim 1$.

$$G(\mathbf{r}, 0) \sim r^{2(y_h - d)} \equiv r^{-(d-2+\eta)}, \qquad \boxed{y_h = \frac{1}{2}(d + 2 - \eta)}$$

- Specific heat

$$c \sim \left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0} \sim |t|^{\frac{d}{y_t}-2} \equiv |t|^{-\alpha}, \qquad \boxed{\alpha = 2 - d\nu}$$

- Spontaneous magnetization

$$m \sim \left.\frac{\partial f}{\partial h}\right|_{h=0} \sim (-t)^{\frac{d}{y_t}-\frac{y_h}{y_t}} \equiv (-t)^{\beta}, \qquad \boxed{\beta = \nu\left(\frac{d-2+\eta}{2}\right)}$$

- Susceptibility

$$\chi \sim \left.\frac{\partial m}{\partial h}\right|_{h=0} \sim |t|^{(d-2y_h)/y_t} \equiv |t|^{-\gamma}, \qquad \boxed{\gamma = \nu(2-\eta)}$$

- m-h curve at $T = T_c$

$$m(h) \sim \frac{\partial f}{\partial h} = |t|^{\frac{d-y_h}{y_t}} \Phi' \left(\frac{h}{|t|^{y_h/y_t}} \right)$$

As $t \to 0$, this must be finite. So we require that $\Phi'(x) \sim x^{d/y_h - 1}$ as $x \to \infty$ so that as $t \to 0$

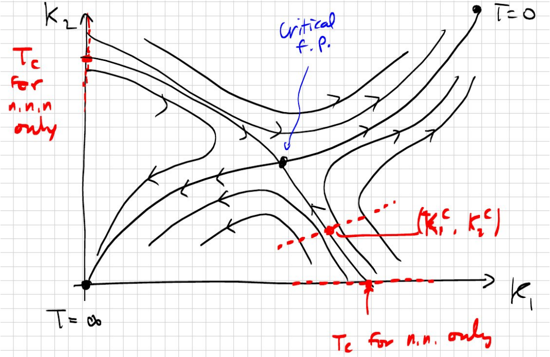
$$m(h) \sim |t|^{\frac{d-y_h}{y_t}} |t|^{(-d/y_h+1)(y_h/y_t)} h^{d/y_h - 1} \sim h^{d/y_h - 1} \equiv h^{1/\delta}$$

$$\boxed{\delta = \frac{d+2-\eta}{d-2+\eta}}$$

example

$$-\mathcal{H} = K_1 \sum_{\text{n.n.}} \sigma_i \sigma_j + K_2 \sum_{\text{n.n.n.}} \sigma_i \sigma_j,$$

where n.n=nearest neighbor and n.n.n=next nearest neighbor.



- Models with $K_2 = 0$ or $K_1 = 0$ or $K_1 \neq 0,\ K_2 \neq 0$ are all in the same universality class.
- Behavior near the critical fixed point controls the critical exponents not initial conditions.

summary

- RG transformation: Successive coarse-graining the short-distance degrees of freedom of the system
- Reveals the information on how the system behaves in the long-distance limit.
- RG flow near critical fixed points plays an important role in determining the critical phenomena of the system.
- More systematic method: Momentum-shell RG
  - Spin $S_i \rightarrow$ Field $\phi(\mathbf{x})$
  - Fourier components: $\tilde{\phi}(\mathbf{q})$
  - Integrate away the field with momentum between $\Lambda/b < q < \Lambda$, where $\Lambda \sim a^{-1}$ (a=lattice spacing)
  - Diagrammatics in Field Theory